This chapter will show you ...

- what similar triangles are
- how to work out the scale factor between similar figures
- how to use the scale factor to work out lengths in similar figures
- how to use the scale factor to work out areas and volumes of similar shapes

What you should already know

- The meaning of congruency
- How to calculate a ratio and cancel it down
- The square and cubes of integers
- How to solve equations of the form \( \frac{x}{y} = \frac{2}{3} \)

Quick check → ANSWERS

1. Which of the following triangles is congruent to this triangle.
   
   - a
   - b
   - c
   - d
   - e

2. Solve the equations.
   
   a. \( \frac{x}{12} = \frac{7}{3} \)
   b. \( \frac{x}{10} = \frac{21}{13} \)
Similar triangles

In this section you will learn how to:
● show two triangles are similar
● work out the scale factor between similar triangles

Key word similar

Triangles are similar if their corresponding angles are equal. Their corresponding sides are then in the same ratio.

EXAMPLE 1

The triangles ABC and PQR are similar. Find the length of the side PR.

Take two pairs of corresponding sides, one pair of which must contain the unknown side. Form each pair into a fraction, so that \( x \) is on top. Since these fractions must be equal

\[
\frac{PR}{AC} = \frac{PQ}{AB}
\]

\[
x = \frac{9}{6}
\]

To find \( x \):

\[
x = \frac{9 \times 6}{6} \text{ cm} \Rightarrow x = \frac{72}{6} = 12 \text{ cm}
\]

EXERCISE 14A

These diagrams are drawn to scale. What is the scale factor of the enlargement in each case?

If you need to revise scale of enlargement, look back at Section 8.5.
Are these pairs of shapes similar? If so, give the scale factor. If not, give a reason.

a

b

c

d

Explain why these shapes are similar.

Give the ratio of the sides.

Which angle corresponds to angle C?

Which side corresponds to side QP?

Explain why these shapes are similar.

Which angle corresponds to angle A?

Which side corresponds to side AC?

Explain why triangle ABC is similar to triangle AQR.

Which angle corresponds to the angle at B?

Which side of triangle AQR corresponds to side AC of triangle ABC?

Your answers to question 4 may help you.

In the diagrams a to f, each pair of shapes are similar but not drawn to scale. Find the lengths of the sides as requested.

Find x.

Find PQ.

Find x and y.

Find x and y.
a. Explain why these two triangles are similar.

b. What is the ratio of their sides?

c. Use Pythagoras’ theorem to calculate the length of side AC of triangle ABC.

d. Write down the length of the side PR of triangle PQR.

A model railway is made to a scale of 1 : 40. If the model bridge is 12 cm high, how high would a real railway bridge be? Give your answer in metres.

**Special cases of similar triangles**

**EXAMPLE 2**

Find the sides marked \(x\) and \(y\) in these triangles (not drawn to scale).

Triangles AED and ABC are similar. So using the corresponding sides CB, DE with AC, AD gives

\[
\frac{x}{5} = \frac{10}{4}
\]

\[x = \frac{10 \times 5}{4} = 12.5\]

Using the corresponding sides AE, AB with AD, AC gives

\[
\frac{y + 6}{6} = \frac{10}{4} \Rightarrow y + 6 = \frac{10 \times 6}{4} = 15
\]

\[y = 15 - 6 = 9\]
**EXAMPLE 3**

Ahmed wants to work out the height of a tall building. He walks 100 paces from the building and sticks a pole, 2 metres long, vertically into the ground. He then walks another 10 paces on the same line and notices that when he looks from ground level, the top of the pole and the top of the building are in line. How tall is the building?

First, draw a diagram of the situation and label it.

Using corresponding sides $ED, CB$ with $AD, AB$ gives

\[
\frac{x}{2} = \frac{110}{10}
\]

\[
\Rightarrow x = \frac{110 \times 2}{10} = 22 \text{ m}
\]

Hence the building is 22 metres high.

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**EXERCISE 14B**

In each of the cases below, state a pair of similar triangles and find the length marked $x$. Separate the similar triangles if it makes it easier for you.

**a**

\[
\frac{4 \text{ cm}}{3 \text{ cm}} = \frac{x}{8 \text{ cm}}
\]

Find $x$. 

**b**

\[
\frac{5 \text{ cm}}{4 \text{ cm}} = \frac{x}{10 \text{ cm}}
\]

Find CE.

---

This diagram shows a method of working out the height of a tower.

A stick, 2 metres long, is placed vertically 120 metres from the base of a tower so that the top of the tower and the top of the stick is in line with a point on the ground 3 metres from the base of the stick. How high is the tower?

It is known that a factory chimney is 330 feet high. Patrick paces out distances as shown in the diagram, so that the top of the chimney and the top of the flag pole are in line with each other. How high is the flag pole?

The shadow of a tree and the shadow of a golf flag coincide, as shown in the diagram. How high is the tree?

Find the height of a pole which casts a shadow of 1.5 metres when at the same time a man of height 165 cm casts a shadow of 75 cm.

Andrew, who is about 120 cm tall, notices that when he stands at the bottom of his garden, which is 20 metres away from his house, his dad, who is about 180 cm tall, looks as big as the house when he is about 2.5 metres away from Andrew. How high is the house?
More complicated problems

The information given in a similar triangle situation can be more complicated than anything you have so far met, and you will need to have good algebraic skills to deal with it. Example 4 is typical of the more complicated problem you may be asked to solve, so follow it through carefully.

EXAMPLE 4

Find the value of $x$ in this triangle.

You know that triangle ABC is similar to triangle ADE.

Splitting up the triangles may help you to see what will be needed.

So your equation will be

$$\frac{x + 15}{x} = \frac{30}{20}$$

Cross multiplying (moving each of the two bottom terms to the opposite side and multiplying) gives

$$20x + 300 = 30x$$

$$\Rightarrow 300 = 10x$$

$$\Rightarrow x = 30 \text{ cm}$$

EXERCISE 14C

Find the lengths $x$ or $x$ and $y$ in the diagrams 1 to 6.
There are relationships between the lengths, areas and volumes of similar shapes.

You saw on pages 182–184 that when a plane shape is enlarged by a given scale factor to form a new, similar shape, the corresponding lengths of the original shape and the new shape are all in the same ratio, which is equal to the scale factor. This scale factor of the lengths is called the length ratio or linear scale factor.

Two similar shapes also have an area ratio, which is equal to the ratio of the squares of their corresponding lengths. The area ratio, or area scale factor, is the square of the length ratio.

Likewise, two 3-D shapes are similar if their corresponding lengths are in the same ratio. Their volume ratio is equal to the ratio of the cubes of their corresponding lengths. The volume ratio, or volume scale factor, is the cube of the length ratio.

Generally, the relationship between similar shapes can be expressed as

- Length ratio \( x : y \)
- Area ratio \( x^2 : y^2 \)
- Volume ratio \( x^3 : y^3 \)

**EXAMPLE 5**

A model yacht is made to a scale of \( \frac{1}{20} \) of the size of the real yacht. The area of the sail of the model is 150 cm\(^2\). What is the area of the sail of the real yacht?

At first sight, it may appear that you do not have enough information to solve this problem, but it can be done as follows.

- Linear scale factor = 1 : 20
- Area scale factor = 1 : 400 (square of the linear scale factor)
- Area of real sail = 400 \times \text{area of model sail}
  = 400 \times 150 \text{ cm}^2
  = 60 000 \text{ cm}^2 = 6 \text{ m}^2
**EXAMPLE 6**

A bottle has a base radius of 4 cm, a height of 15 cm and a capacity of 650 cm$^3$. A similar bottle has a base radius of 3 cm.

a What is the length ratio?

b What is the volume ratio?

c What is the volume of the smaller bottle?

a The length ratio is given by the ratio of the two radii, that is 4 : 3.

b The volume ratio is therefore $4^3 : 3^3 = 64 : 27$.

Let $v$ be the volume of the smaller bottle. Then the volume ratio is

\[
\frac{\text{Volume of smaller bottle}}{\text{Volume of larger bottle}} = \frac{v}{650} = \frac{27}{64}
\]

\[\Rightarrow v = \frac{27 \times 650}{64} = 274 \text{ cm}^3 \text{ (3 significant figures)}\]

**EXAMPLE 7**

The cost of a paint can, height 20 cm, is £2.00 and its label has an area of 24 cm$^2$.

a If the cost is based on the amount of paint in the can, what is the cost of a similar can, 30 cm high?

b Assuming the labels are similar, what will be the area of the label on the larger can?

a The cost of the paint is proportional to the volume of the can.

Length ratio $= 20 : 30 = 2 : 3$

Volume ratio $= 2^3 : 3^3 = 8 : 27$

Let $P$ be the cost of the larger can. Then the cost ratio is

\[
\frac{\text{Cost of larger can}}{\text{Cost of smaller can}} = \frac{P}{2}
\]

Therefore,

\[\frac{P}{2} = \frac{27}{8}\]

\[\Rightarrow P = \frac{27 \times 2}{8} = £6.75\]

b Area ratio $= 2^2 : 3^2 = 4 : 9$

Let $A$ be the area of the larger label. Then the area ratio is

\[
\frac{\text{Larger label area}}{\text{Smaller label area}} = \frac{A}{24}
\]

Therefore,

\[\frac{A}{24} = \frac{9}{4}\]

\[\Rightarrow A = \frac{9 \times 24}{4} = 54 \text{ cm}^2\]
The length ratio between two similar solids is 2 : 5.

a What is the area ratio between the solids?

b What is the volume ratio between the solids?

The length ratio between two similar solids is 4 : 7.

a What is the area ratio between the solids?

b What is the volume ratio between the solids?

Copy and complete this table.

<table>
<thead>
<tr>
<th>Linear scale factor</th>
<th>Linear ratio</th>
<th>Linear fraction</th>
<th>Area scale factor</th>
<th>Volume scale factor</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1 : 2</td>
<td>1/1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4 : 1</td>
<td>1/4</td>
<td></td>
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<tr>
<td>1/3</td>
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<td>25</td>
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<td>1/125</td>
</tr>
<tr>
<td>1 : 7</td>
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<td></td>
<td>1/100</td>
<td></td>
</tr>
<tr>
<td>5 : 1</td>
<td></td>
<td></td>
<td>1/25</td>
<td></td>
</tr>
</tbody>
</table>

Some years ago, a famous beer advertisement showed a bar attendant taking an ordinary pint glass and filling it with beer underneath the counter. When the glass reappeared, it was full of beer and its width and height were twice those of the original glass. The slogan on the advertisement was “The pint that thinks it’s a quart”. (A quart is 2 pints.)

a What was the length ratio of the two glasses used in the advertisement?

b What was the volume ratio of the two glasses?

c The smaller glass held a pint. How much would the larger glass have held?

d Is the advertisement fair?

A shape has an area of 15 cm². What is the area of a similar shape whose lengths are three times the corresponding lengths of the first shape?

A toy brick has a surface area of 14 cm². What would be the surface area of a similar toy brick whose lengths are?

a twice the corresponding lengths of the first brick?

b three times the corresponding lengths of the first brick?

A sheepskin rug covers 12 ft² of floor. What area would be covered by rugs with these lengths?

a twice the corresponding lengths of the first rug

b half the corresponding lengths of the first rug
A brick has a volume of 300 cm\(^3\). What would be the volume of a similar brick whose lengths are

a. twice the corresponding lengths of the first brick?

b. three times the corresponding lengths of the first brick?

 Thirty cubic centimetres of clay were used to make a model sheep. What volume of clay would be needed to make a similar model sheep with these lengths?

a. five times the corresponding lengths of the first model

b. one half of the corresponding lengths of the first model

A can of paint, 6 cm high, holds a half a litre of paint. How much paint would go into a similar can which is 12 cm high?

It takes 1 litre of paint to fill a can of height 10 cm. How much paint does it take to fill a similar can of height 45 cm?

It takes 1.5 litres of paint to fill a can of height 12 cm.

a. How much paint does it take to fill a similar can whose dimensions are 1\(\frac{1}{2}\) times the corresponding dimensions of the first can?

b. Which of the information given is not needed to be able to answer part a?

To make a certain dress, it took 2.4 m\(^2\) of material. How much material would a similar dress need if its lengths were

a. 1.5 times the corresponding lengths of the first dress?

b. three quarters of the corresponding lengths of the first dress?

A model statue is 10 cm high and has a volume of 100 cm\(^3\). The real statue is 2.4 m high. What is the volume of the real statue? Give your answer in m\(^3\).

A small can of paint costs 75p. What is the cost of a larger similar can whose circumference is twice that of the smaller can? Assume that the cost is based only on the volume of paint in the can.

A triangle has sides of 3, 4 and 5 cm. Its area is 6 cm\(^2\). How long are the sides of a similar triangle that has an area of 24 cm\(^2\)?

A ball with a radius of \(r\) cm has a volume of 10 cm\(^3\). What is the radius of a ball with a volume of 270 cm\(^3\)?
Calculate the area of each of the shaded faces and hence calculate the volume of each of these solids. (They are not drawn to scale.)

Using area and volume ratios

In some problems involving similar shapes, the length ratio is not given, so we have to start with the area ratio or the volume ratio. We usually then need first to find the length ratio in order to proceed with the solution.

**Example 8**

A manufacturer makes a range of clown hats that are all similar in shape. The smallest hat is 8 cm tall and uses 180 cm² of card. What will be the height of a hat made from 300 cm² of card?

The area ratio is 180 : 300

Therefore, the length ratio is $\sqrt{180} : \sqrt{300}$ (do not calculate these yet)

Let the height of the larger hat be $H$, then

$$\frac{H}{8} = \frac{\sqrt{300}}{\sqrt{180}} = \frac{\sqrt{300}}{\sqrt{180}}$$

$$\Rightarrow H = 8 \times \sqrt{\frac{300}{180}} = 10.3 \text{ cm (1 decimal place)}$$
EXAMPLE 9

A supermarket stocks similar small and large cans of soup. The areas of their labels are 110 cm\(^2\) and 190 cm\(^2\) respectively. The weight of a small can is 450 g. What is the weight of a large can?

The area ratio is 110 : 190

Therefore, the length ratio is \(\sqrt[3]{110} : \sqrt[3]{190}\) (do not calculate these yet)

So the volume (weight) ratio is \((\sqrt[3]{110})^3 : (\sqrt[3]{190})^3\).

Let the weight of a large can be \(W\), then

\[
\frac{W}{450} = \left(\frac{\sqrt[3]{190}}{\sqrt[3]{110}}\right)^3 = \left(\frac{190}{110}\right)^3
\]

\[\Rightarrow W = 450 \times \left(\frac{190}{110}\right)^3 = 1020 \text{ g} \quad (3 \text{ significant figures})
\]

EXAMPLE 10

Two similar cans hold respectively 1.5 litres and 2.5 litres of paint. The area of the label on the smaller can is 85 cm\(^2\). What is the area of the label on the larger can?

The volume ratio is 1.5 : 2.5

Therefore, the length ratio is \(\sqrt[3]{1.5} : \sqrt[3]{2.5}\) (do not calculate these yet)

So the area ratio is \((\sqrt[3]{1.5})^2 : (\sqrt[3]{2.5})^2\)

Let the area of the label on the larger can be \(A\), then

\[
\frac{A}{85} = \left(\frac{\sqrt[3]{2.5}}{\sqrt[3]{1.5}}\right)^2 = \left(\frac{2.5}{1.5}\right)^2
\]

\[\Rightarrow A = 85 \times \left(\frac{2.5}{1.5}\right)^2 = 119 \text{ cm}^2 \quad (3 \text{ significant figures})
\]

EXERCISE 14E

1. A firm produces three sizes of similarly shaped labels for its products. Their areas are 150 cm\(^2\), 250 cm\(^2\) and 400 cm\(^2\). The 250 cm\(^2\) label just fits around a can of height 8 cm. Find the heights of similar cans around which the other two labels would just fit.

2. A firm makes similar gift boxes in three different sizes: small, medium and large. The areas of their lids are as follows.

   - small: 30 cm\(^2\),
   - medium: 50 cm\(^2\),
   - large: 75 cm\(^2\)

   The medium box is 5.5 cm high. Find the heights of the other two sizes.
3. A cone, height 8 cm, can be made from a piece of card with an area of 140 cm². What is the height of a similar cone made from a similar piece of card with an area of 200 cm²?

4. It takes 5.6 litres of paint to paint a chimney which is 3 m high. What is the tallest similar chimney that can be painted with 8 litres of paint?

5. A man takes 45 minutes to mow a lawn 25 m long. How long would it take him to mow a similar lawn only 15 m long?

6. A piece of card, 1200 cm² in area, will make a tube 13 cm long. What is the length of a similar tube made from a similar piece of card with an area of 500 cm²?

7. All television screens (of the same style) are similar. If a screen of area 220 cm² has a diagonal length of 21 cm, what will be the diagonal length of a screen of area 350 cm²?

8. Two similar statues, made from the same bronze, are placed in a school. One weighs 300 g, the other weighs 2 kg. The height of the smaller statue is 9 cm. What is the height of the larger statue?

9. A supermarket sells similar cans of pasta rings in three different sizes: small, medium and large. The sizes of the labels around the cans are as follows.
   - small can: 24 cm²
   - medium can: 46 cm²
   - large can: 78 cm²

   The medium size can is 6 cm tall with a weight of 380 g. Calculate these quantities.
   a. the heights of the other two sizes
   b. the weights of the other two sizes

10. Two similar bottles are 20 cm and 14 cm high. The smaller bottle holds 850 ml. Find the capacity of the larger one.

11. A statue weighs 840 kg. A similar statue was made out of the same material but two fifths the height of the first one. What was the weight of the smaller statue?

12. A model stands on a base of area 12 cm². A smaller but similar model, made of the same material, stands on a base of area 7.5 cm². Calculate the weight of the smaller model if the larger one is 3.5 kg.

13. A solid silver statue was melted down to make 100 000 similar miniatures, each 2 cm high. How tall was the original statue?

14. Two similar models have volumes 12 m³ and 30 m³. If the surface area of one of them is 2.4 m², what are the possible surface areas of the other model?
Two rectangles have the dimensions shown.

Are the rectangles similar? Explain your answer clearly.

Triangle ABC is similar to triangle CDE. Calculate the length of CD.

In the triangle PQR, AB is parallel to QR. AB = 10 cm, QR = 16 cm and BR = 12 cm. Find the length PB.

Two cylinders, P and Q, are mathematically similar. The total surface area of cylinder P is 90π cm². The total surface area of cylinder Q is 810π cm². The length of cylinder P is 4 cm.

a) Work out the length of cylinder Q.

b) Work out the volume of cylinder Q. Give your answer as a multiple of π.

X and Y are two geometrically similar solid shapes. The total surface area of shape X is 450 cm². The total surface area of shape Y is 800 cm². The volume of shape X is 1350 cm³. Calculate the volume of shape Y.

PQ and PXY are similar triangles. Calculate the length of RY.

WORKED EXAM QUESTION

A camping gas container is in the shape of a cylinder with a hemispherical top. The dimensions of the container are shown in the diagram.

It is decided to increase the volume of the container by 20%. The new container is mathematically similar to the old one. Calculate the base diameter of the new container.

Solution

Old Volume : New volume = 100% : 120% = 1 : 1.2

\[ \sqrt[3]{1 : 1.2} = 1 : 1.06265 \]

New diameter = Old diameter x 1.06265 = 8 x 1.06265 = 8.5 cm

First find the volume scale factor.

Take the cube root to get the linear scale factor.

Multiply the old diameter by the linear scale factor to get the new diameter.
Martin works for a light company called “Bright Ideas”. He has been asked to calculate accurate measurements for a new table lamp the company are going to produce. The three main components are the base, the stem and the shade. Below is a sketch of the side view, and an “exploded” diagram which shows the lamp in more detail.

The base and stem are made from a material which has a density of 10 g/cm³. Help Martin complete the table to find the total weight of the stem and the base.

<table>
<thead>
<tr>
<th>volume of stem</th>
<th>cm³</th>
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<tbody>
<tr>
<td>weight of stem</td>
<td>g</td>
</tr>
<tr>
<td>volume of base</td>
<td>cm³</td>
</tr>
<tr>
<td>weight of base</td>
<td>g</td>
</tr>
<tr>
<td>total weight</td>
<td>g</td>
</tr>
</tbody>
</table>

The lampshade is the frustum of a cone. It is to be made from a fire-proof material. Martin draws a sketch showing the dimensions he knows. Help him to calculate the missing dimensions and then the surface area of the shade.

<table>
<thead>
<tr>
<th>length of X</th>
<th>cm</th>
</tr>
</thead>
<tbody>
<tr>
<td>length of L</td>
<td>cm</td>
</tr>
<tr>
<td>length of L</td>
<td>cm</td>
</tr>
<tr>
<td>surface area of small cone</td>
<td>cm²</td>
</tr>
<tr>
<td>surface area of large cone</td>
<td>cm²</td>
</tr>
<tr>
<td>surface area of lampshade</td>
<td>cm²</td>
</tr>
</tbody>
</table>
Martin knows a cone is made from the sector of a circle, but he needs to calculate the angle of the sector $\theta$. He draws this diagram of the large cone to help.

Find the angle $\theta$ for Martin.

A material trim is to go around the bottom and top circles of the lampshade. Help Martin use this diagram to calculate the total length of trim needed.

Draw an accurate scale drawing for Martin of the side view of the lamp using a scale of 3 : 1.
GRADE YOURSELF

Know why two shapes are similar
Able to work out unknown sides using scale factors and ratios
Able to set up equations to find missing sides in similar triangles
Able to solve problems using area and volume scale factors
Able to solve practical problems using similar triangles
Able to solve related problems involving, for example, capacity, using area and volume scale factors

What you should know now

● How to find the ratios between two similar shapes
● How to work out unknown lengths, areas and volumes of similar 3-D shapes
● How to solve practical problems using similar shapes
● How to solve problems using area and volume ratios